Functions for the General Solution of Parametric Word Equations

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Abstract:

In this article we introduce the functions $Fi(x_1, x_2)^{\lambda_1,\dots,\lambda_s}$ *and* $Th(x_1, x_2, x_3)_i^{\lambda_1,\dots,\lambda_{2s}}$ *(i = 1, 2, 3), of the word variables* x_i *and of the natural number variables* λ_i*, where* $s \geq 0$*. By means of these functions, we give exactly the general solution (i.e. the set of all the solutions) of the first basic parametric equation:* $x_1x_2x_3x_4 = x_3x_1^{\lambda}x_2x_5$*, in a free monoid.*

1. Introduction

The following four parametric equations:

$x_1x_2x_3x_4 = x_3x_1^{\lambda}x_2x_5$,

$x_1x_2x_3x_4 = x_2x_3^{\lambda}x_1x_5$,

$x_1x_2^2x_3x_4 = x_3x_1^2x_2x_5$,

$x_1x_2^{\lambda+1}x_3x_4 = x_3x_2^{\mu+1}x_1x_5$,

in a free monoid, are called *basic* equations. They arise in the graph of the prefixe-equations in free monoid (cf. [2], [3]) and play an important role in the hierarchy of the parametric equations, in reason of the structures of their solutions. In particular, the general solution of any equation in a free monoid of the form $\Phi(x_1, x_2, x_3)\, x_4 = \Psi(x_1, x_2, x_3)\, x_5$, where $\Phi(x_1, x_2, x_3)$ and $\Psi(x_1, x_2, x_3)$ are any words on the alphabet $\{x_1, x_2, x_3\}$, is described by means of a finite number of parametric transformations and by the general solutions of the basic equations.

2. Definitions and notations

Let Π be a free monoid (a free semigroup with unit) with a countable alphabet of generators (called also constants):

$$a_1, a_2, \dots, a_k, \dots \quad \textbf{(1)}$$

Let

$$x_1, x_2, \dots, x_n, \dots \quad \textbf{(2)}$$

be a countable alphabet of word variables, and let

$$\lambda_1, \lambda_2, \dots, \lambda_i, \dots \quad \textbf{(3)}$$

be a countable alphabet of natural number variables (called also *natural parameters*).

We denote by ϕ the set of all linear polynomials in naturals, of the form:

$k_0 + \sum_{i=1}^{r} k_i \lambda_i$, where $r, k_0, k_1, \dots, k_r$ are natural numbers, and $\lambda_1, \dots, \lambda_r$ are natural parameters.

Define inductively a *parametric word* as follows:

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— Any word on the alphabet (2) is a parametric word.

— If P is a parametric word, L is a linear polynomial in naturals, then $(P)^L$ is a parametric word.

— If P and Q are two parametric words, then PQ is a parametric word.

Every letter of the alphabet (2), and every parametric word of the form $(P)^L$, where L is a linear polynomial in naturals, is called a *parametric letter*. The parametric word P is called the *base* of the parametric letter $(P)^L$.

By definition, two parametric words are *equal* if any one can be obtained from the other by means of the equalities:

$$(P)^0 = 1,\ (P)^1 = P,\ (P)^{L_1+L_2} = (P)^{L_1}(P)^{L_2},$$

where P is a parametric word, L_1, L_2 are linear polynomials in naturals.

The parametric letters $(P_1)^{L_1+L_3}$ and $(P_2)^{L_1+L_3}$ are called *left cancellable* if P_1 equals P_2, and $L_3 \neq 0$.

A *parametric equation* in a free monoid is given by a left noncancellable equality of parametric words

$$\Phi(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_t) = \Psi(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_t) \qquad \textbf{(4)}$$

If Φ and Ψ are empty words, the equation (4) is called the *trivial* equation, denoted by **1**.

A parametric transformation is defined by the application:

$$\begin{cases} x_1 \to W_1(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_q) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ x_n \to W_n(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_q) \\ \lambda_1 \to L_1(\lambda_1, \ldots, \lambda_q) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \lambda_t \to L_t(\lambda_1, \ldots, \lambda_q) \end{cases} \qquad \textbf{(5)}$$

where W_i is a parametric word, and L_i is a linear polynomial in naturals. The components of the form $x_i \to x_i$, and $\lambda_i \to \lambda_i$ are usually omitted.

The result of the application of the transformation (5) to the parametric equation (4) is the parametric equation given by the substitution of (5) and all possible left cancellations.

A parametric transformation (5) is called a *parametric solution* of the equation (4), if the result of the application of the transformation (5) to the equation (4) is the trivial equation.

A *coefficient transformation* is defined by the application:

$$\begin{cases} x_1 \to X_1 \\ \ldots\ldots\ldots \\ x_n \to X_n \\ \lambda_1 \to \Lambda_1 \\ \ldots\ldots\ldots \\ \lambda_t \to \Lambda_t \end{cases} \qquad \textbf{(6)}$$

where X_i are words on the alphabet (1), and Λ_i are natural numbers. The components of the form $x_i \to 1$ and $\lambda_i \to 0$ are usually omitted.

A coefficient transformation (6) is *applicable* to the parametric equation (4) if the words $\Phi(X_1, \ldots, X_n, \Lambda_1, \ldots, \Lambda_t)$ and $\Psi(X_1, \ldots, X_n, \Lambda_1, \ldots, \Lambda_t)$ coincide. By definition, the result of this application is the trivial equation.

A coefficient transformation (6) is called a *solution* of the equation (4), if the words $\Phi(X_1, \ldots, X_n, \Lambda_1, \ldots, \Lambda_t)$ and $\Psi(X_1, \ldots, X_n, \Lambda_1, \ldots, \Lambda_t)$ coincide.

A coefficient transformation

$$\begin{cases} x_1 \to X_1 \\ \dots\dots\dots \\ x_n \to X_n \\ \lambda_1 \to \Lambda_1 \\ \dots\dots\dots \\ \lambda_p \to \Lambda_p \end{cases}$$

is called an *extension* of the coefficient transformation (6), if $p \geq t$.

We will say that the parametric transformation T *contains* the coefficient transformation C, by means of the coefficient transformation C_1 if $TC_1 = C$.

A list of parametric solutions of the equation E will be called a *general solution* of E, if every solution of E is contained in some parametric solution of this list. The general solution of E will be denoted <E>.

The length of a word A on the alphabet (1) will be denoted $|A|$.

Let us define parametric solutions which satisfy *conditions on the lengths of solutions*, and that which satisfy *conditions on natural parameters*.

A *condition on natural parameters* has the form:

$$L_1(\lambda_1, \dots, \lambda_q) \#_i\ L_2(\lambda_1, \dots, \lambda_q) \qquad \textbf{(7)}$$

where $\#_i \in \{<, \leq, =\}$, and L_1, L_2 are linear polynomials in naturals. A coefficient transformation (6) is called a *solution* of the equation E with a condition (7) on natural parameters, if (6) is a solution of E that satisfies:

$L_1(\Lambda_1,\dots, \Lambda_q)) \#_i\ L_2(\Lambda_1,\dots, \Lambda_q)$, where $\#_i$ is the relation determined by (7).

A *condition on the length of a solution* has the form:

$$\partial(P_1(x_1,\dots, x_n, \lambda_1,\dots, \lambda_q)) \#_i\ \partial(P_2(\xi_1,\dots, x_n, \lambda_1,\dots, \lambda_q)) \qquad \textbf{(8)}$$

where $\#_i \in \{<, \leq, =\}$, and P_1, P_2 are two parametric words. A coefficient transformation (6) is called a *solution* of the equation E with a condition (8) on the lengths of solutions, if (6) is a solution of E that satisfies:

$|P_1(X_1, \dots, X_n, \Lambda_1,\dots, \Lambda_q)| \#_i\ |P_2(X_1, \dots, X_n, \Lambda_1,\dots, \Lambda_q)|$, where $\#_i$ is the relation determined by (8).

Let E be an equation with conditions, and let $R_1, \dots, R_m$ be the list of new conditions. By (E, R_i), we denote the equation E with the condition R_i. The equation E is said to be *divided into a collection of equations with conditions* (E, R_1), …, (E, R_m), if every solution S of E is a solution of some (E, R_i).

We say that the equation E_1 is r*educed by the parametric transformation T to the equation E_2* if $E_1T = E_2$, and for every solution S_1 of the equation E_1 there exists an extension S_2* of a solution S_2 of the equation E_2 such that $S_1 = TS_2$*. We say that S_2* is the *image* of S_1 obtained by T. Note that the need of the extension S_2* is justified by the fact that T could have some variables that are not in E_2.

Lemma 1:

Let E_1 be reduced by T to E_2. And let S_1 be a solution of E_1 and S_2* its *image* obtained by T. Then we have:

— If Q_2 is a parametric solution of E_2, then TQ_2 is a parametric solution of E_1.

— If the parametric solution Q_2 contains a solution S_2 of E_2, then the parametric solution TQ_2 of E_1 contains the solution S_1 of E_1.

♦

Theorem 1:

Let the equation E_1 be reduced by the parametric transformation T to the equation E_2. If the general solution <E_2> of E_2 is $Q_1, \dots, Q_r, \dots$ then the general solution <E_1> of E_1 is $TQ_1, \dots, TQ_r, \dots$

♦

Theorem 2:

Let the equation E be divided into a collection of equations with conditions (E, R_1), …, (E, R_m), with the list of conditions $R_1, \dots, R_m$. And let the general solution

$<(E, R_i)>$ of (E, R_i) be $Q_{i,1}, \ldots, Q_{i,r_1}, \ldots$ Then, the general solution $<E>$ of E is $Q_{1,1}, \ldots, Q_{1,r_1}, \ldots, Q_{m,1}, \ldots, \ldots, Q_{m,r_m} \ldots$

♦

3. The function $Fi(x_1, x_2)^{\lambda 1, \ldots, \lambda s}$

Define inductively the function $Fi(x_1, x_2)^{\lambda 1, \ldots, \lambda s}$, for $s \geq 0$, x_1 and x_2 are two word variables, and $\lambda 1, \ldots, \lambda s$ are natural parameters.

$Fi(x_1, x_2) = 1.$

$Fi(x_1, x_2)^{\lambda 1, \lambda 2 \ldots, \lambda s} = (Fi(x_2, x_1)^{\lambda 2 \ldots, \lambda s}\, x_2)^{\lambda 1}\, Fi(x_1, x_2)^{\lambda 3, \ldots, \lambda s}$; $\quad (s \geq 1)$.

In particular:

$Fi(x_1, x_2)^{\lambda 1} = (x_2)^{\lambda 1}$

$Fi(x_1, x_2)^{\lambda 1, \lambda 2} = ((x_1)^{\lambda 2}\, x_2)^{\lambda 1}$

$Fi(x_1, x_2)^{\lambda 1, \lambda 2, \lambda 3} = (((x_2)^{\lambda 3}\, x_1)^{\lambda 2}\, x_2)^{\lambda 1}\, (x_2)^{\lambda 3}$.

Theorem Fi1:

The following identities hold for $k \geq 0$.

$Fi(x_1, x_2)^{\lambda 1, \ldots, \lambda 2k} = Fi(x_1, x_1^{\lambda 2k}\, x_2)^{\lambda 1, \ldots, \lambda 2k-1}$.

$Fi(x_1, x_2)^{\lambda 1, \ldots, \lambda 2k+1} = Fi(x_2^{\lambda 2k+1}\, x_1, x_2)^{\lambda 1, \ldots, \lambda 2k}\, x_2^{\lambda 2k+1}$.

♦

Consider an equation of the form $x_1 P x_2 \ldots = x_2 Q x_1 \ldots$, where P, Q are words on the alphabet $x_3, \ldots, x_n, \ldots$ and consider the sequence of parametric transformations: **(9)**

1. $x_1 \to (x_2\, Q)^{\lambda_1}\, x_1$
2. $x_2 \to (x_1\, P)^{\lambda_2}\, x_2$
3. $x_1 \to (x_2\, Q)^{\lambda_3}\, x_1$
4. $x_2 \to (x_1\, P)^{\lambda_4}\, x_2$

…………………………….

2k-1. $x_1 \to (x_2\, Q)^{\lambda_{2k-1}}\, x_1$

2k. $x_2 \to (x_1\, P)^{\lambda_{2k}}\, x_2$

2k+1. $x_1 \to (x_2\, Q)^{\lambda_{2k+1}}\, x_1$

…………………………….

where $\lambda_1, \lambda_2, \ldots$ are natural parameters.

Theorem Fi2:

For every natural s, the sequence (9) of the first s parametric transformation can be collected by the following common transformation:

$$\begin{cases} x_1 \to Fi(x_1 P\, , x_2 Q)^{\lambda 1, \lambda 2 \ldots, \lambda s}\, x_1 \\ x_2 \to Fi(x_2 Q\, , x_1 P)^{\lambda 2 \ldots, \lambda s}\, x_2 \end{cases}$$

♦

3. The function $Th(x_1, x_2, x_3)_i^{\lambda 1, \lambda 2 \ldots, \lambda 2s}$

Define, by a joint induction, the functions $Th(x_1, x_2, x_3)_1^{\lambda 1, \lambda 2 \ldots, \lambda 2s}$, $Th(x_1, x_2, x_3)_2^{\lambda 1, \lambda 2 \ldots, \lambda 2s}$, $Th(x_1, x_2, x_3)_3^{\lambda 1, \lambda 2 \ldots, \lambda 2s}$, for $s \geq 0$, x_1, x_2 and x_3 are three word variables, and $\lambda 1, \lambda 2 \ldots, \lambda 2s$ are natural parameters.

$Th(x_1, x_2, x_3)_i = 1 \quad (i = 1,2,3).$

$Th(x_1, x_2, x_3)_1^{\lambda 1, \lambda 2 \ldots, \lambda 2s} =$

$(Th(x_1, x_2, x_3)_3^{\lambda 1, \lambda 2\ldots, \lambda 2s} x_3)^{\lambda 1} Th(x_1, x_2, x_3)_1^{\lambda 3, \ldots, \lambda 2s}$.

$Th(x_1, x_2, x_3)_2^{\lambda 1, \lambda 2\ldots, \lambda 2s} =$

$(Th(x_1, x_2, x_3)_2^{\lambda 3, \ldots, \lambda 2s} x_2 Th(x_1, x_2, x_3)_3^{\lambda 3, \ldots, \lambda 2s} x_3 Th(x_1, x_2, x_3)_1^{\lambda 3, \ldots, \lambda 2s} x_1 Th(x_1, x_2, x_3)_1^{\lambda 1, \lambda 2\ldots, \lambda 2s} x_1)^{\lambda 2} Th(x_1, x_2, x_3)_2^{\lambda 3, \ldots, \lambda 2s}$.

$Th(x_1, x_2, x_3)_3^{\lambda 1, \lambda 2\ldots, \lambda 2s} =$

$Th(x_1, x_2, x_3)_1^{\lambda 3, \ldots, \lambda 2s} x_1 Th(x_1, x_2, x_3)_2^{\lambda 3, \ldots, \lambda 2s} x_2 Th(x_1, x_2, x_3)_3^{\lambda 3, \ldots, \lambda 2s}$.

And define inductively the auxiliary function $Oc(x_1, x_2, x_3)^{\lambda 1, \lambda 2\ldots, \lambda 2s}$ for $s \geq 0$.

$Oc(x_1, x_2, x_3) = 1$.

$Oc(x_1, x_2, x_3)^{\lambda 1, \lambda 2\ldots, \lambda 2s} =$

$Oc(x_1, x_2, x_3)^{\lambda 3\ldots, \lambda 2s} (Th(x_1, x_2, x_3)_3^{\lambda 3, \ldots, \lambda 2s} x_3 Th(x_1, x_2, x_3)_1^{\lambda 3, \ldots, \lambda 2s} x_1 Th(x_1, x_2, x_3)_2^{\lambda 3, \ldots, \lambda 2s} x_2)^{\lambda 1}$.

<u>Theorem Th_1:</u>

The following identities hold for $s \geq 0$:

$Th(x_1, x_2, x_3)_1^{\lambda 1, \lambda 2\ldots, \lambda 2s+2} =$

$= Th((x_1x_2x_3)^{\lambda 2s+1} x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda 2s+1}x_1)^{\lambda 2s+2}x_2, x_1x_2x_3)_1^{\lambda 1,\lambda 2\ldots, \lambda 2s}(x_1x_2x_3)^{\lambda 2s+1}$

$Th(x_1, x_2, x_3)_2^{\lambda 1, \lambda 2\ldots, \lambda 2s+2} =$

$= Th((x_1x_2x_3)^{\lambda 2s+1} x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda 2s+1}x_1)^{\lambda 2s+2}x_2, x_1x_2x_3)_2^{\lambda 1,\lambda 2\ldots, \lambda 2s}(x_2x_3x_1(x_1x_2x_3)^{\lambda 2s+1}x_1)^{\lambda 2s+2}$

$Th(x_1, x_2, x_3)_3^{\lambda 1, \lambda 2\ldots, \lambda 2s+2} =$

$= Th((x_1x_2x_3)^{\lambda 2s+1} x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda 2s+1}x_1)^{\lambda 2s+2}x_2, x_1x_2x_3)_3^{\lambda 1,\lambda 2\ldots, \lambda 2s}x_1x_2$

$Oc(x_1, x_2, x_3)^{\lambda 1, \lambda 2\ldots, \lambda 2s+2} =$

$= (x_3x_1x_2)^{\lambda 2s+1} Oc((x_1x_2x_3)^{\lambda 2s+1} x_1, (x_2x_3x_1(x_1x_2x_3)^{\lambda 2s+1})^{\lambda 2s+2}x_2, x_1x_2x_3)^{\lambda 1,\lambda 2\ldots, \lambda 2s}$

♦

Consider the sequence of joint parametric transformations: **(10)**

$$\begin{cases} x_1 \to (x_1x_2x_3)^{\lambda_1} x_1 \\ x_2 \to (x_2x_3x_1(x_1x_2x_3)^{\lambda 1}x_1)^{\lambda_2} x_2 \\ x_3 \to x_1x_2x_3 \\ x_4 \to x_4 (x_3x_1x_2)^{\lambda_1} \end{cases} \quad \mathbf{(10.1)}$$

$$\begin{cases} x_1 \to (x_1x_2x_3)^{\lambda_3} x_1 \\ x_2 \to (x_2x_3x_1(x_1x_2x_3)^{\lambda 3}x_1)^{\lambda_4} x_2 \\ x_3 \to x_1x_2x_3 \\ x_4 \to x_4 (x_3x_1x_2)^{\lambda_3} \end{cases} \quad \mathbf{(10.2)}$$

……………………………

$$\begin{cases} x_1 \to (x_1x_2x_3)^{\lambda 2s-1}\, x_1 \\ x_2 \to (x_2x_3x_1(x_1x_2x_3)^{\lambda 2s-1}x_1)^{\lambda 2s}\, x_2 \\ x_3 \to x_1x_2x_3 \\ x_4 \to x_4\, (x_3x_1x_2)^{\lambda 2s-1} \end{cases} \qquad \mathbf{(10.s)}$$

$$\begin{cases} x_1 \to (x_1x_2x_3)^{\lambda 2s+1}\, x_1 \\ x_2 \to (x_2x_3x_1(x_1x_2x_3)^{\lambda 2s+1}x_1)^{\lambda 2s+2}\, x_2 \\ x_3 \to x_1x_2x_3 \\ x_4 \to x_4\, (x_3x_1x_2)^{\lambda 2s+1} \end{cases} \qquad \mathbf{(10.s+1)}$$

……………………………

Theorem $^{\mathbf{Th}}$2:

For every natural s, the sequence of the s joint parametric transformations (10) can be collected by the following common transformation:

$$\begin{cases} x_1 \to {}^{Th}(x_1, x_2, x_3)_1{}^{\lambda 1, \lambda 2\ldots, \lambda 2s}\, x_1 \\ x_2 \to {}^{Th}(x_1, x_2, x_3)_2{}^{\lambda 1, \lambda 2\ldots, \lambda 2s}\, x_2 \\ x_3 \to {}^{Th}(x_1, x_2, x_3)_3{}^{\lambda 1, \lambda 2\ldots, \lambda 2s}\, x_3 \\ x_4 \to x_4{}^{Oc}(x_1, x_2, x_3)^{\lambda 1, \lambda 2\ldots, \lambda 2s} \end{cases} \qquad (^{\mathbf{Th}})$$

♦

4. Equations and Solutions

Proposition 1:

The general solution of the equation

$$x_1x_2 = x_2x_1 \qquad \mathbf{(A)}$$

is described by the transformation:

$$\begin{cases} x_1 \to x_1{}^{\alpha} \\ x_2 \to x_1{}^{\beta} \end{cases}$$

where α and β are natural parameters.

♦

Proposition 2:

The general solution of the equation

$$x_1x_2x_3 = x_3x_1x_2 \qquad \mathbf{(B)}$$

is described by the transformations:

$$\begin{cases} x_1 \to 1 \\ x_2 \to 1 \end{cases} \qquad \begin{cases} x_1 \to (x_1x_2)^{\alpha}x_1 \\ x_2 \to (x_2x_1)^{\beta}x_2 \\ x_3 \to (x_1x_2)^{\gamma} \end{cases}$$

where α, β and γ are natural parameters.

♦

Proposition 3:

The general solution of the equation

$$x_1x_3 = x_2{}^{\alpha}x_1 \qquad \mathbf{(C)}$$

where α is natural parameter, is described by the transformations:

t) $\begin{cases} x_2 \to 1 \\ x_3 \to 1 \end{cases}$ **tt)** $\begin{cases} x_3 \to 1 \\ \alpha \to 0 \end{cases}$ **ttt)** $\begin{cases} x_1 \to & (x_1x_2)^{\beta}x_1 \\ x_2 \to & x_1x_2 \\ x_3 \to & (x_2x_1)^{\alpha} \end{cases}$

where β is a natural parameter.

♦

Proposition 4:

The general solution of the equation

$$x_1x_2x_3x_4 = x_2(x_1x_2)^{\beta+1}x_3 \qquad \textbf{(D)}$$

where β is a natural parameter, is described by the transformations:

t) $\begin{cases} x_1 \to & x_1^{\gamma} \\ x_2 \to & x_1^{\delta} \end{cases}$

< (C) with $\alpha = \delta + (\gamma + \delta)\,\beta$>.

where γ and δ are natural parameters.

♦

By $Fi(x_1, x_2)^{\lambda1, \lambda2\ldots, \lambda\sigma}$, where σ is a natural parameter, we denote the function depending on σ, whose values are the functions $Fi(x_1, x_2)$, $Fi(x_1, x_2)^{\lambda1}$, $Fi(x_1, x_2)^{\lambda1, \lambda2}$, ...

Proposition 5:

The general solution of the equation

$$x_1x_3x_2 = x_2x_1x_4, \text{ with } \partial(x_2) > 0,\ \partial(x_3) > 0 \qquad \textbf{(E)}$$

is described by the transformations:

t) $\begin{cases} x_1 \to & Fi(x_1x_3, x_2)^{\lambda1, \lambda2\ldots, \lambda\sigma}x_1 \\ x_2 \to & Fi(x_2, x_1x_3)^{\lambda2\ldots, \lambda\sigma}x_2 \end{cases}$

$x_2 \to x_1x_2$

$x_3 \to x_2x_3$

$x_4 \to x_4x_2$

< (C) with $\alpha = 1$>

♦

By $Th(x_1, x_2, x_3)_i^{\lambda1, \lambda2\ldots, \lambda2\sigma}$, where σ is a natural parameter, we denote the function depending on σ, whose values are the functions $Th(x_1, x_2, x_3)_i$, $Th(x_1, x_2, x_3)_i^{\lambda1, \lambda2}$, $Th(x_1, x_2, x_3)_i^{\lambda1, \lambda2, \lambda3, \lambda4}$, ...

Proposition 6:

The general solution of the equation

$$x_1x_2x_3x_4 = x_3x_1^2x_2, \text{ with } \partial(x_3) > 0 \qquad \textbf{(F)}$$

is described by the transformations **r)** :

$$\begin{cases} x_1 \to Th(x_1, x_2, x_3)_1^{\lambda1, \lambda2\ldots, \lambda2\sigma}\, x_1 \\ x_2 \to Th(x_1, x_2, x_3)_2^{\lambda1, \lambda2\ldots, \lambda2\sigma}\, x_2 \\ x_3 \to Th(x_1, x_2, x_3)_3^{\lambda1, \lambda2\ldots, \lambda2\sigma}\, x_3 \\ x_4 \to x_4Oc(x_1, x_2, x_3)^{\lambda1, \lambda2\ldots, \lambda2\sigma} \end{cases}$$

$x_1 \to x_3^{\lambda2\sigma+1}x_1,$

$x_3 \rightarrow \; x_1x_3$

$x_2 \rightarrow \; (x_3x_1{}^2(x_3x_1)^{\lambda_{2\sigma+1}})^{\lambda_{2\sigma+2}}x_2$

followed by one of the four transformations t), tt), ttt), tttt), where:

t) $x_2 \rightarrow \; x_3x_2$

$x_1 \rightarrow \; x_2x_1$

$x_4 \rightarrow \; x_4x_2(x_1x_3x_2)^{\lambda_{2\sigma+1}}$

< (D) with $\alpha = 0$>

tt) $x_2 \rightarrow \; x_3x_1(x_1x_3)^{\gamma}x_2$, with $\gamma < \lambda_{2\sigma+1}$

$x_1 \rightarrow \; x_2x_1$

$x_4 \rightarrow \; (x_1x_3x_2)^{\lambda_{2\sigma+1}-\gamma}\, x_1x_2(x_1x_3x_2)^{\gamma}$

< (B) >

ttt) $x_2 \rightarrow \; x_3x_1(x_1x_3)^{\gamma}x_1x_2$, with $\gamma < \lambda_{2\sigma+1}$

$x_3 \rightarrow \; x_2x_3$

$x_4 \rightarrow \; (x_3x_1x_2)^{\lambda_{2\sigma+1}-\gamma-1}\, x_3x_1{}^2x_2(x_1x_3x_2)^{\gamma}$

< (B) >

tttt) $x_2 \rightarrow \; x_3x_1(x_1x_3)^{\lambda_{2\sigma+1}}x_2$

$x_1 \rightarrow \; x_2x_1$

$x_4 \rightarrow \; x_1x_2(x_1x_3x_2)^{\lambda_{2\sigma+1}}$

< (B) >

♦

Proposition 7:

The general solution of the equation

$$x_1x_2x_3x_4 = x_3x_2(x_2(x_3x_2)^{\lambda+1})^{\mu+1}x_1, \text{ with } \partial(x_2x_3) > 0 \qquad \textbf{(G)}$$

where λ, μ are natural parameters is described by the transformations:

$x_1 \rightarrow \; (x_3x_2(x_2(x_3x_2)^{\lambda+1})^{\mu+1})^{\rho}x_1$, where ρ is a natural parameter,

followed by one of the five transformations t), tt), ttt), tttt), ttttt) where:

t) $x_3 \rightarrow \; x_1x_3$

$x_4 \rightarrow \; x_4x_3x_2(x_1x_3x_2)^{\lambda}(x_2(x_1x_3x_2)^{\lambda+1})^{\mu}x_1$

<(F)>

tt) $x_1 \rightarrow \; x_3x_1$

$x_2 \rightarrow \; x_1x_2$

$x_4 \rightarrow \; x_4x_1x_2(x_3x_1x_2)^{\lambda}(x_1x_2(x_3x_1x_2)^{\lambda+1})^{q}x_3x_1$

<(D) with $\alpha = 0$>

ttt) $x_1 \rightarrow \; x_3x_2(x_2(x_3x_2)^{\lambda+1})^{\tau}x_1$, with $\tau < \mu+1$.

$x_2 \rightarrow \; x_1x_2$

$x_4 \rightarrow \; x_2(x_3x_1x_2)^{\lambda}(x_1x_2(x_3x_1x_2)^{\lambda+1})^{\mu-\tau}x_3x_1x_2(x_1x_2(x_3x_1x_2)^{\lambda+1})^{\tau}x_1$

<(B) >

tttt) $x_1 \rightarrow \; x_3x_2(x_2(x_3x_2)^{\lambda+1})^{\tau}x_2(x_3x_2)^{\upsilon}x_1$ with $\tau < \mu+1$, $\upsilon < \lambda+1$.

$x_3 \rightarrow \; x_1x_3$

followed by one of the three transformations tttt1), tttt2), tttt3) where:

tttt1) $x_4 \rightarrow x_3x_2(x_1x_3x_2)^{\lambda-\upsilon-1}(x_2(x_1x_3x_2)^{\lambda+1})^{\mu-\tau}x_1x_3x_2(x_2(x_1x_3x_2)^{\lambda+1})^{\tau}x_2(x_1x_3x_2)^{\upsilon}x_1$, with $\lambda>\upsilon$.
<(B)>

tttt2) $x_4 \rightarrow x_3x_2(x_2(x_1x_3x_2)^{\lambda+1})^{\tau}x_2(x_1x_3x_2)^{\upsilon}x_1$, with $\lambda = \upsilon$, $\mu = \tau$.
<(B)>

tttt3) $x_4 \rightarrow x_4x_3x_2(x_1x_3x_2)^{\lambda}(x_2(x_1x_3x_2)^{\lambda+1})^{\mu-\tau-1}x_1x_3x_2(x_2(x_1x_3x_2)^{\lambda+1})^{\tau}x_2(x_1x_3x_2)^{\lambda}x_1$, with $\lambda = \upsilon$, $\mu > \tau$.
followed by the transformation <(F)> or <(C) with $\alpha = 0$>.

ttttt) $x_1 \rightarrow x_3x_2(x_2(x_3x_2)^{\lambda+1})^{\tau}x_2(x_3x_2)^{\upsilon}x_3x_1$ with $\tau< \mu+1$, $\upsilon< \lambda+1$.

$x_2 \rightarrow x_1x_2$
followed by one of the three transformations ttttt1), ttttt2), ttttt3) where:

ttttt1) $x_4 \rightarrow x_2(x_3x_1x_2)^{\lambda-\upsilon-1}(x_1x_2(x_3x_1x_2)^{\lambda+1})^{\mu-\tau}x_3x_1x_2(x_1x_2(x_3x_1x_2)^{\lambda+1})^{\tau}x_1x_2(x_3x_1x_2)^{\upsilon}x_3x_1$, with $\lambda>\upsilon$.
<(B)>

ttttt2) $x_4 \rightarrow x_2(x_1x_2(x_3x_1x_2)^{\lambda+1})^{\tau}x_1x_2(x_3x_1x_2)^{\upsilon}x_3x_1$, with $\lambda = \upsilon$, $\mu = \tau$.
<(B)>

ttttt3) $x_4 \rightarrow x_1x_2(x_3x_1x_2)^{\lambda}(x_1x_2(x_3x_1x_2)^{\lambda+1})^{\mu-\tau-1}x_3x_1x_2(x_1x_2(x_3x_1x_2)^{\lambda+1})^{\tau}x_1x_2(x_3x_1x_2)^{\lambda}x_3x_1$
<(D) with $\alpha = 0$, $\lambda = \upsilon$, $\mu > \tau$>.

♦

__**Proposition 8:**__

The general solution of the equation

$$x_1x_2x_3x_4 = (x_3x_1)^{\mu+2}x_2 \text{ with } \partial(x_3) > 0 \qquad \textbf{(H)}$$

where μ is a natural parameter is described by the transformations:

$x_1 \rightarrow x_3^{\lambda}x_1$

$x_3 \rightarrow x_1x_3$
< (G)>.

♦

__**Proposition 9:**__

The general solution of the equation

$$x_1x_2x_3x_4 = x_2(x_1x_2)^{\tau}x_3x_1\Psi(x_1, x_2, x_3) \qquad \textbf{(I)}$$

where τ is a natural parameter, $\Psi(x_1, x_2, x_3)$ is a parametric word, is described by the transformations:

t) $\tau \rightarrow \alpha + 1$

$x_4 \rightarrow x_4x_1\Psi(x_1, x_2, x_3)$
<(D)>

tt) $\tau \rightarrow 0$

$x_4 \rightarrow \Psi(x_1, x_2, x_3)$,
<(B)>

♦

__**Proposition 10:**__

The general solution of the equation

$$x_1x_2x_3x_4 = x_3x_2^{\mu+2}x_1 \text{ with } \partial(x_2) > 0 \qquad \textbf{(J)}$$

where μ is a natural parameter is described by the transformations:

$$\begin{cases} x_1 \to Fi(x_1x_2,\ x_3x_2^{\mu+2})^{\lambda 1,\ \lambda 2\ldots,\ \lambda s}x_1 \\ x_3 \to Fi(x_3x_2^{\mu+2},\ x_1x_2)^{\lambda 2\ldots,\ \lambda s}x_3 \end{cases}$$

followed by one of the two transformations t), tt) where:

t) $x_3 \to x_1x_3$

$x_2 \to x_3x_2$

<(H)>

tt) $x_1 \to x_3x_2^{\upsilon}x_1 \quad (\upsilon < \mu + 2)$

$x_2 \to x_1x_2$

<(I)>

♦

Proposition 11:

The general solution of the equation

$$x_1x_2x_3x_4 = x_2^{\mu+3}x_3x_1 \text{ with } \partial(x_2) > 0 \qquad \textbf{(K)}$$

where μ is a natural parameter is described by the transformations:

t) $x_1 \to x_2^{\mu+3}x_1$

<J>

tt) $x_1 \to x_2^{\upsilon}x_1 \ (\upsilon < \mu + 3)$

$x_2 \to x_1x_2$

<I>

♦

Proposition 12:

The general solution of the equation

$$x_1x_2x_3x_4 = x_3x_1^{\mu+3}x_2 \qquad \textbf{(L)}$$

where μ is a natural parameter, is described by the transformations:

t) $x_1 \to 1$

$x_4 \to 1$

<(A)>

tt) $x_3 \to (x_1x_2)^{\lambda}$

<(C) with $\alpha = \mu + 2$>

ttt) $x_3 \to (x_1x_2)^{\lambda}\ x_3$

$x_1 \to x_3x_1$

<(H)>

tttt) $x_3 \to (x_1x_2)^{\lambda}\ x_1\ x_3$

$x_2 \to x_3x_2$

<(K)>

♦

Proposition 13:

The general solution of the basic equation

$$x_1x_2x_3x_4 = x_3x_1^{\lambda}x_2x_5 \qquad \textbf{(M)}$$

where λ is a natural parameter, is described by the transformations:

t1) $\lambda \to 0$

$x_1 \to x_3x_1$

$x_5 \to \quad x_5\, x_3 x_4$

<(C) with $\alpha = 1$>

t2) $\lambda \to \quad 0$

$x_3 \to \quad x_1 x_3$

$x_5 \to \quad x_5\, x_4$

<(E) >

t3) $\lambda \to \quad 0$

$x_1 \to \quad 1,$

$x_5 \to \quad x_4,$

<(A)>

t4) $\lambda \to \quad 1$

$x_5 \to \quad x_4,$

<(B)>

t5) $\lambda \to \quad 2$

$x_4 \to \quad x_4 x_5$

<(F)>

t6) $\lambda \to \quad 2$

$x_3 \to \quad 1,$

$x_4 \to \quad x_4 x_5$

<(C) with $\alpha = 1$>

t7) $\lambda \to \quad \mu + 3$

$x_4 \to \quad x_4 x_5$

<(L)>

♦

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